UNIVERSAL COLLECTIVE ROTATION CHANNELS AND QUANTUM ERROR CORRECTION

MARIUS JUNGE[1], PETER T. KIM[2], DAVID W. KRIBS[2,3,4]

ABSTRACT. We present and investigate a new class of quantum channels, what we call 'universal collective rotation channels', that includes the class of collective rotation channels as a special case. The fixed point set and noise commutant coincide for a channel in this class. Computing the precise structure of this C*-algebra is a core problem in a particular noiseless subsystem method of quantum error correction. We prove that there is an abundance of noiseless subsystems for every channel in this class and that the Young tableaux combinatorial machine may be used to explicitly compute these subsystems.

1. INTRODUCTION

The study of quantum channels is a central theme in quantum computing and quantum information theory [**31**]. A fundamental class of quantum channels is known as the class of *collective rotation channels* [**4, 5, 6, 12, 13, 16, 17, 20, 22, 36, 37, 38, 39, 41**]. This class has its roots in the postulates of quantum mechanics and has recently played a key role in experimental efforts towards realizing certain quantum error correction methods [**13, 37**]. Of particular interest in the current study is the *noise commutant method of noiseless subsystems*. This is a recently developed paradigm for passive quantum error correction [**11, 13, 17, 21, 22, 30, 40**]. In this method, the 'noise commutant' is used as a vehicle for encoding states that are left immune to the noise of a given channel. The operator algebras generated by such states are called 'noiseless subsystems'.

In this paper, we present a new class of quantum channels and investigate them in the context of quantum error correction, with specific

reference to the noiseless subsystem method. This class is a generalization of the collective rotation class, which arises as an important special case, hence we use the appellation 'universal collective rotation channels' to describe this class. We prove that the noise commutant for every channel in this class has rich structure and hence contains an abundance of noiseless subsystems. To accomplish this, we use operator algebra techniques to make an explicit connection with representation theory of the symmetric group and, as a consequence, the Young tableaux combinatorial machine **[14, 18]** may be used to explicitly compute these noiseless subsystems.

The paper is organized as follows. Section 2 contains introductory material on quantum channels and quantum error correction. In Section 3 we define and establish basic properties of the class of universal collective rotation (*ucr*-) channels. We make the connection with representation theory of the symmetric group in Section 4 and show that the noise commutant for ucr-channels is determined by a particular representation of the symmetric group. In Sections 5 and 6 we collect well-known facts from representation theory of the symmetric group, with emphasis on Young tableaux combinatorics, and work through some low-dimensional examples. We finish with a concluding remark in Section 7 and discuss possible avenues of further research.

2. Quantum Channels and Noiseless Subsystems

Let $\mathcal{H}$ be a (complex) Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the set of bounded operators on $\mathcal{H}$. When a basis for $\mathcal{H}$ is fixed and $\dim \mathcal{H} = k < \infty$, the algebra $\mathcal{B}(\mathcal{H})$ may be identified with the set of all complex $k \times k$ matrices $\mathbb{M}_k = \mathbb{M}_k(\mathbb{C})$. Throughout the paper, if we are given positive integers $n \geq 1$ and $d \geq 2$, we let $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$ be a fixed orthonormal basis for $d-$dimensional Hilbert space $\mathcal{H}_d = \mathbb{C}^d$ and

let $\{|i_1 i_2 \cdots i_n\rangle : i_j \in \mathbb{Z}_d\}$ be the corresponding orthonormal basis for $\mathcal{H}_{d^n} = (\mathbb{C}^d)^{\otimes n}$.

A linear map $\mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is *completely positive* if for all $k \geq 1$ the ampliation maps $\mathbb{1}_k \otimes \mathcal{E} : \mathbb{M}_k \otimes \mathcal{B}(\mathcal{H}) \to \mathbb{M}_k \otimes \mathcal{B}(\mathcal{H})$ are positive. See [**25, 32**] for introductions to the study of completely positive maps from different perspectives. A *quantum channel* is a map $\mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ that is completely positive and trace preserving. Given $\mathcal{E}$, there is ([**8, 26**]) a set of *noise operators*, or *errors*, $\{E_k\}$ on $\mathcal{H}$ such that

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger \quad \text{for} \quad \rho \in \mathcal{B}(\mathcal{H}). \tag{1}$$

Trace preservation means that the noise operators satisfy

$$\sum_k E_k^\dagger E_k = \mathbb{1},$$

where $\mathbb{1}$ is the identity operator on $\mathcal{H}$. The channel is unital if also,

$$\mathcal{E}(\mathbb{1}) = \sum_k E_k E_k^\dagger = \mathbb{1}.$$

We will denote the fixed point set for $\mathcal{E}$ by

$$\mathrm{Fix}(\mathcal{E}) = \{\rho \in \mathcal{B}(\mathcal{H}) : \mathcal{E}(\rho) = \rho\}.$$

Further let $\mathcal{A}$ be the algebra generated by $\{E_k\}$ from (1). This is called the *interaction algebra* in quantum computing [**22**]. It is a relic of the channel in the sense that the same algebra is obtained whatever the choice of noise operators in (1). This is most succinctly seen in the case of a unital channel. In general, $\mathrm{Fix}(\mathcal{E})$ is just a $\dagger$-closed subspace of $\mathcal{B}(\mathcal{H})$, but in the case of a unital channel $\mathcal{E}$, the so-called *noise commutant*

$$\mathcal{A}' = \{\rho \in \mathcal{B}(\mathcal{H}) : \rho E_k = E_k \rho, \ \forall k\}$$

coincides with the fixed point set [**7, 28**]:

$$\mathrm{Fix}(\mathcal{E}) = \mathcal{A}'.$$

In particular, $\mathrm{Fix}(\mathcal{E}) = \mathcal{A}'$ is a †-closed operator algebra (a finite dimensional C*-algebra [**2, 10, 35**]). In this case the von Neumann double commutant theorem from operator algebras shows how the algebra $\mathcal{A} = \mathcal{A}'' = \mathrm{Fix}(\mathcal{E})'$ only depends on the channel.

Every finite dimensional C*-algebra is unitarily equivalent to an orthogonal direct sum of 'ampliated' full matrix algebras; *i.e.*, there is a unitary operator U such that

$$U\mathcal{A}U^\dagger = \bigoplus_{k=1}^{r} \big(\mathbb{1}_{m_k} \otimes \mathbb{M}_{n_k}\big).$$

From the representation theory perspective, a factor $\mathbb{1}_{m_k} \otimes \mathbb{M}_{n_k}$ corresponds to an n_k-dimensional irreducible representation appearing with multiplicity m_k. With this form for $\mathcal{A}$ given, the structure of the commutant up to unitary equivalence is easily computed by

$$U\,\mathrm{Fix}(\mathcal{E})U^\dagger = U\mathcal{A}'U^\dagger = \bigoplus_{k=1}^{r} \big(\mathbb{M}_{m_k} \otimes \mathbb{1}_{n_k}\big). \tag{2}$$

(See [**13, 17, 20, 36, 38, 39**] for more detailed discussions in connection with quantum information theory.)

Given a quantum channel $\mathcal{E}$ with noise operators $\{E_k\}$, the structure of the noise commutant $\mathcal{A}'$ can be used to prepare density operators for use in the noiseless subsystem method of error correction. This is a passive method of quantum error correction, in the sense that such operators will remain immune to the effects of the noise of the channel, without active intervention. Thus, computing the precise structure of $\mathcal{A}'$ as in (2) is of fundamental importance in this method. We mention that for experimental reasons [**29**], only one matrix algebra $\mathbb{M}_{m_k} \otimes \mathbb{1}_{n_k}$ may be used at a time in this manner. Hence it is also desirable to find the largest full matrix algebra which is a subalgebra of the noise commutant.

3. Universal Collective Rotation Channels

For the rest of the paper, given a positive integer $d \geq 2$ we write $\mathbb{M}_d$ for the operator algebra $\mathcal{B}(\mathbb{C}^d)$ represented as $d \times d$ complex matrices with respect to the standard basis $\{|0\rangle, \ldots, |d-1\rangle\}$ for $\mathbb{C}^d$. Further let $\mathbb{M}_{d,sa}$ be the subset of self-adjoint matrices inside $\mathbb{M}_d$.

Fix $n \geq 1$. Given $1 \leq k \leq n$ we define a representation of $\mathbb{M}_d$ on $\mathcal{H}_{d^n}$ by

$$\omega_k(x) = \mathbb{1}_d \otimes \cdots \otimes \mathbb{1}_d \otimes \underbrace{x}_{k\text{-th position}} \otimes \mathbb{1}_d \otimes \cdots \otimes \mathbb{1}_d$$

for all $x \in \mathbb{M}_d$. Then we may define sums of independent copies of x by

$$u_n(x) = \sum_{k=1}^{n} \omega_k(x) \quad \text{for} \quad x \in \mathbb{M}_d.$$

Definition 3.1. Given a finite subset $\mathcal{S} \subset \mathbb{M}_{d,sa}$, we define a *universal collective rotation (ucr-) channel* $\mathcal{E}_{\mathcal{S}}$ by

$$\mathcal{E}_{\mathcal{S}}(\rho) = \frac{1}{\sqrt{|\mathcal{S}|}} \sum_{x \in \mathcal{S}} e^{i\theta_x u_n(x)} \ \rho \ e^{-i\theta_x u_n(x)} \quad \text{for} \quad \rho \in \mathcal{B}(\mathcal{H}_{d^n}),$$

where $\{\theta_x : x \in \mathcal{S}\}$ are non-zero angles.

Given a set of operators $\mathcal{R}$, define $\operatorname{Alg}\mathcal{R}$ to be the operator algebra generated by $\mathcal{R}$. This is the set of all polynomials in the elements of $\mathcal{R}$. When $\mathcal{R}$ is a self-adjoint set, $\operatorname{Alg}\mathcal{R}$ is a C*-algebra. Through a standard functional calculus argument from operator theory, it follows that the interaction algebra $\mathcal{A}_{\mathcal{S}}$ for $\mathcal{E}_{\mathcal{S}}$ is obtained as $\mathcal{A}_{\mathcal{S}} \equiv \operatorname{Alg}\{e^{i\theta_x u_n(x)} : x \in \mathcal{S}\} = \operatorname{Alg}\{u_n(x) : x \in \mathcal{S}\}$. Thus by von Neumann's double-commutant identity we have

$$\mathcal{A}_{\mathcal{S}} = \{e^{i\theta_x u_n(x)} : x \in \mathcal{S}\}'' = \{u_n(x) : x \in \mathcal{S}\}''.$$

Notice that $\mathcal{A}_{\mathcal{S}}$ is independent of the choice of (non-zero) angles θ_x. As an application of the fixed point theorem from **[7, 28]** we obtain the following.

Theorem 3.2. *If $\mathcal{S}$ is a finite subset of $\mathbb{M}_{d,sa}$, then the ucr-channel $\mathcal{E}_{\mathcal{S}}$ satisfies*

$$\mathrm{Fix}(\mathcal{E}_{\mathcal{S}}) = \mathcal{A}'_{\mathcal{S}}.$$

Observe that $\mathbb{1}_d$ belongs to $\mathcal{A}_{\mathcal{S}}$ from its characterization as a bicommutant. Since $u_n(\mathbb{1}_d) = n\mathbb{1}_{d^n}$, we may always add $\mathbb{1}_d$ to $\mathcal{S}$ without changing the properties of $\mathrm{Fix}(\mathcal{E}_S)$. This motivates the following definition.

Definition 3.3. We will say that $\mathcal{S}$ is *maximal* if $\mathrm{span}\{x : x \in \mathcal{S}\}$ contains all matrices $x \in \mathbb{M}_d$ with $\mathrm{tr}(x) = 0$.

It turns out that for maximal $\mathcal{S}$ the algebra $\mathcal{A}_{\mathcal{S}}$ is a well-known object in representation theory.

Remark 3.4. To place the class of ucr-channels in context, we note that the ucr-channels for $d = 2$ and general n are the class of 'two-level' collective rotation channels from quantum computing [**4, 5, 6, 12, 13, 16, 17, 20, 22, 36, 37, 38, 39, 41**]. The noise operators in this case are also denoted by J_x, J_y, J_z and they arise in quantum mechanics as the canonical representation of the angular momentum relations [**9**]. From the noiseless subsystem/quantum error correction perspective, the algebra $\mathrm{Fix}(\mathcal{E}_{\mathcal{S}}) = \mathcal{A}'_{\mathcal{S}}$ for this subclass of ucr-channels, and natural d-dimensional representations of the J_k operators, has been analyzed in [**16**] from an operator theory cum quantum mechanics point of view.

4. Representation Theory and the Noise Commutant

In this section we identify the structure of the noise commutant in terms of representation theory for the symmetric group. We begin with some notation. We shall denote the n-fold tensor product of $\mathbb{M}_d$ by

$$\mathbb{M}_d^{\otimes n} = \underbrace{\mathbb{M}_d \otimes \cdots \otimes \mathbb{M}_d}_{n\text{-times}} \cong \mathbb{M}_{d^n}.$$

Let $\mathrm{Sym}^n\mathbb{M}_d$ be the subalgebra of $\mathbb{M}_{d^n}$ generated by the symmetric tensor products; that is, $\mathrm{Sym}^n\mathbb{M}_d$ is the algebra generated by the operators

$$\Phi_n(x_1 \otimes \cdots \otimes x_n) = \frac{1}{n!}\sum_{\pi \in S_n} x_{\pi(1)} \otimes \cdots \otimes x_{\pi(n)},$$

where each $x_i \in \mathbb{M}_d$ and S_n is the permutation group on n letters.

In terms of representation theory, we may equally well consider the representation $\pi : \mathrm{GL}(d) \to \mathrm{GL}(d^n)$ given by $\pi(u) = u \otimes \cdots \otimes u$, and then we have

$$\mathrm{Sym}^n\mathbb{M}_d = \pi(\mathrm{GL}(d))'',$$

where $\mathrm{GL}(d)$ is the group of $d \times d$ nonsingular complex matrices. This tensor product representation of $\mathrm{GL}(d)$ is in 'duality' with the representation of the symmetric group S_n defined on vector tensors by

$$\pi(\sigma)(h_1 \otimes \cdots \otimes h_n) = h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(n)},$$

for $\sigma \in S_n$ and $h_1, \ldots, h_n \in \mathcal{H}_d$. In this context, Schur's classical duality theorem reads as

Theorem 4.1. $\pi(S_n)' = \mathrm{Sym}^n\mathbb{M}_d$.

We use the following characterization of $\mathrm{Sym}^n\mathbb{M}_d$ below.

Lemma 4.2. *For positive integers d and n, we have*

$$\mathrm{Sym}^n\mathbb{M}_d = \{x^{\otimes n} : x \in \mathbb{M}_d\}'' = \{u_n(x) : x \in \mathbb{M}_d\}''$$

Proof. It is clear that $\mathrm{Sym}^n\mathbb{M}_d$ contains the C*-algebra

$$\begin{aligned} \mathcal{B} = \{x^{\otimes n} : x \in \mathbb{M}_d\}'' &= \mathrm{span}\{x^{\otimes n} : x \in \mathbb{M}_d\} \\ &= \mathrm{Alg}\{x^{\otimes n} : x \in \mathbb{M}_d\} \end{aligned}$$

as a subalgebra. For the converse inclusion, let $x_1, \ldots, x_n \in \mathbb{M}_d$ and consider the complex matrix integral

$$\begin{aligned}
&\int_{z_1,\ldots,z_n\in\mathbb{T}} \Big(\sum_{j=1}^n z_j x_j\Big)^{\otimes n} \frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n} \\
&= \sum_{j_1,\ldots,j_n=1}^n \left(\int_{z_1,\ldots,z_n\in\mathbb{T}} \prod_{r=1}^n z_{j_r} \frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}\right)(x_{j_1}\otimes\cdots\otimes x_{j_n}) \\
&= \sum_{j_1,\ldots,j_n=1}^n \left(\prod_{s=1}^n \int_{z\in\mathbb{T}} z^{|\{r:j_r=s\}|}\frac{dz}{z}\right)(x_{j_1}\otimes\cdots\otimes x_{j_n}) \\
&= (2\pi i)^n \sum_{\pi\in S_n} x_{\pi(1)}\otimes\cdots\otimes x_{\pi(n)},
\end{aligned}$$

where $\mathbb{T}$ denotes the unit circle in the complex plane. It follows that $\Phi_n(x_1\otimes\cdots\otimes x_n)$ belongs to $\mathcal{B}$ for any choice of $x_1,\ldots,x_n$, and hence $\mathcal{B}$ coincides with $\mathrm{Sym}^n\mathbb{M}_d$.

On the other hand, it is clear by definition that $\mathrm{Sym}^n\mathbb{M}_d$ contains the algebra $\{u_n(x) : x \in \mathbb{M}_d\}''$ generated by the $u_n(x)$. Moreover, a consideration of the expansion for $u_n(x)^n$ shows that $x^{\otimes n}$ belongs to this double commutant for all $x \in \mathbb{M}_d$. For the sake of brevity let us observe this fact for $n = 2$ and $n = 3$:

$$\begin{aligned}
x\otimes x &= \frac{1}{2!}\Big(u_2(x)^2 - u_2(x^2)\Big) \\
x\otimes x\otimes x &= \frac{1}{3!}\Big(u_3(x)^3 - 3u_3(x^2)u_3(x) - 2u_3(x^3)\Big).
\end{aligned}$$

In fact, for all $x \in \mathbb{M}_d$, the tensor product $x^{\otimes n}$ belongs to the algebra $\mathrm{Alg}\{u_n(x^p) : 1 \le p \le n\}$. Thus the second characterization of $\mathrm{Sym}^n\mathbb{M}_d$ follows. ■

Observe that as a consequence of this proof, we also have $\mathrm{Sym}^n\mathbb{M}_d = \{u_n(x) : x \in \mathbb{M}_{d,sa}\}''$. We can now explicitly link the noise commutant for these channels with representation theory of the symmetric group.

Theorem 4.3. *Let $\mathcal{S} \subset \mathbb{M}_{d,sa}$ be a maximal system, then*

$$\mathrm{Fix}(\mathcal{E}_{\mathcal{S}}) = \mathcal{A}'_{\mathcal{S}} = \pi(S_n)''.$$

Moreover, for an arbitrary finite set $\mathcal{S} \subset \mathbb{M}_{d,sa}$, we have

$$\mathrm{Fix}(\mathcal{E}_{\mathcal{S}}) \supseteq \pi(S_n)''.$$

Proof. If $\mathcal{S}$ is maximal, then the interaction algebra $\mathrm{Fix}(\mathcal{E}_{\mathcal{S}})' = \mathcal{A}_{\mathcal{S}} = \{u_n(x) : x \in \mathcal{S}\}'' = \{u_n(x) : x \in \mathbb{M}_d\}''$ coincides with $\pi(S_n)'$ by Lemma 4.2. For the second assertion, a given finite subset $\mathcal{S} \subset \mathbb{M}_{d,sa}$ is contained inside a maximal system $\mathcal{S}_{\max}$. Hence $\mathcal{A}_{\mathcal{S}} \subseteq \mathcal{A}_{\mathcal{S}_{\max}}$ and

$$\mathrm{Fix}(\mathcal{E}_{\mathcal{S}}) = \mathcal{A}'_{\mathcal{S}} \supseteq \mathcal{A}'_{\mathcal{S}_{\max}} = \pi(S_n)''.$$

■

5. Computing Noiseless Subsystems Via Young Tableaux

In this section, we collect well-known facts from the representation theory of S_n that allow us to describe $\mathrm{Fix}(\mathcal{E}_{\mathcal{S}}) = \pi(S_n)''$ in an explicit manner. Recall that this is imperative for using the structure of the noise commutant to produce noiseless subsystems.

For the discussion in this section, we shall fix positive integers $d \geq 2$ and $n \geq 2$. Let $\{|0\rangle, \dots, |d-1\rangle\}$ be the orthonormal basis for $\mathcal{H}_d$ corresponding to a given d-level quantum system, and let

$$\big\{|i_1 \cdots i_n\rangle : 0 \leq i_j < d,\ 1 \leq j \leq n\big\}$$

be the corresponding basis for $\mathcal{H}_{d^n}$. Observe that the set of n-tuples $\{i_1, \dots, i_n\}$ is in one-to-one correspondence with the set of functions $f : \{1, \dots, n\} \to \{0, \dots, d-1\}$. So we may define functions k_l for $0 \leq l < d$ by

$$k_l(i_1, ..., i_n) = \#\{1 \leq j \leq n \mid i_j = l\} \quad \text{for} \quad 0 \leq i_j < d,$$

and we have $\sum_{l=0}^{d-1} k_l(i_1, ..., i_n) = n$.

Now, given positive integers $k_0, \dots, k_{d-1}$ with each $0 \leq k_l \leq n$, we define a corresponding subspace of $\mathcal{H}_{d^n}$ by

$$\mathcal{H}_{k_0,\dots,k_{d-1}} = \mathrm{span}\,\big\{|i_1 \cdots i_n\rangle : k_l(i_1, ..., i_n) = k_l,\ 0 \leq l < d\big\}.$$

Notice that $\mathcal{H}_{d^n} = \bigoplus \mathcal{H}_{k_0,\ldots,k_{d-1}}$, where the direct sum runs over all choices of $k_0,\ldots,k_{d-1}$. Clearly, $\mathcal{H}_{k_0,\ldots,k_{d-1}}$ is an invariant (hence reducing) subspace for the action of the symmetric group S_n. More importantly, the irreducible subspaces, or equivalently the decomposition factors of $\mathcal{H}_{k_0,\ldots,k_{d-1}}$ are completely characterized. The key ingredient in this characterization is the notion of Young tableaux.

Given $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$, a non-increasing sequence of positive integers with $\sum_i \lambda_i = n$, put $\lambda = (\lambda_1,\ldots,\lambda_r)$. Then the associated λ-*diagram* is defined as

$$[\lambda] = \{c_{ij} : 1 \leq i \leq r,\, 1 \leq j \leq \lambda_i\},$$

where c_{ij} denotes a 'cell' in $[\lambda]$. Simply put, $[\lambda]$ is a diagram with d rows of cells which are left justified and λ_i cells in the ith row. A λ-*tableau* is a bijective function $t : [\lambda] \to \{1,\ldots,n\}$. Clearly, S_n acts by composition $\sigma t = \sigma \circ t$ on λ-tableaux. Given a λ-tableau, the *column stabilizer* C_t is the subgroup of S_n which leaves the columns of λ setwise fixed. Similarly, the *row stabilizer* R_t is the subgroup of S_n which leaves the rows of λ setwise fixed. Two tableaux t_1 and t_2 are *equivalent* if there exists a permutation $\sigma \in R_{t_1}$ such that $\sigma t_1 = t_2$. In particular, this means that the set of *tabloids* $\mathrm{Tab}_\lambda = \{\{t\} : t \text{ a } \lambda\text{-tableau}\}$ of equivalence classes is indexed by all partitions $(A_1,...,A_r)$ of $\{1,\ldots,n\}$ such that the cardinalities $|A_1| = \lambda_1,\ldots,|A_r| = \lambda_r$. Given a λ-diagram $[\lambda]$, consider the (i,j)-cell c_{ij} in $[\lambda]$. The *hook length* $h(i,j)$ for c_{ij} is the number of cells directly below c_{ij} in the jth column of $[\lambda]$ plus the number of cells to the right of c_{ij} in the ith row of $[\lambda]$ plus one (for the cell c_{ij} itself). Formally,

$$h(i,j) = \lambda_i + \lambda'_j + 1 - i - j,$$

where λ'_j is the number of elements in the j-th column. Also recall that a tableau $t : [\lambda] \to \{1,\ldots,n\}$ is *standard* if the numbers increase along rows and increase down columns. The abstract S_n-module that has an

orthonormal basis in bijective correspondence with elements of Tab_λ is denoted by M^λ, so that

$$M^\lambda = \mathrm{span}\{e_{\{t\}} : \{t\} \in \mathrm{Tab}_\lambda\}.$$

The *Specht module* $\mathcal{S}^\lambda$ is the submodule of M^λ generated by the 'polytabloids'

$$e_t \equiv \sum_{\sigma \in C_t} (\mathrm{sgn}\,\sigma) e_{\sigma\{t\}} \in M^\lambda.$$

Let us summarize the following facts (see chapter 7 in [**18**]).

Theorem 5.1. *Let $k_0, \ldots, k_{d-1}$ be positive integers and consider a partition of $\{1, \ldots, n\}$ into sets $A_0, \ldots, A_{d-1}$ with $|A_l| = k_l$. Let λ be the non-increasing rearrangement of $(k_0, \ldots, k_{d-1})$. Then $\mathcal{H}_{k_0,\ldots,k_{d-1}}$ is isomorphic as an S_n-module to M^λ.*

Every polytabloid e_t is a cyclic vector for the irreducible module $\mathcal{S}^\lambda$. The dimension of $\mathcal{S}^\lambda$ is given by the 'hook length formula'

$$\dim \mathcal{S}^\lambda = \frac{n!}{\prod \textit{hook lengths in } [\lambda]},$$

and a basis for $\mathcal{S}^\lambda$ is given by

$$\{e_t : t \textit{ standard } \lambda\textit{-tableau } \}.$$

Finally, every finite dimensional irreducible representation of S_n is unitarily equivalent to a Specht module representation π_λ, where π_λ is the representation of S_n on $\mathcal{S}^\lambda$ defined by $\pi_\lambda(\sigma)e_t \equiv e_{\sigma t}$.

For the next discussion let us fix numbers $(k_0, \ldots, k_{d-1})$ and let us denote by $\mu = (\mu_0, \ldots, \mu_{d-1})$ the non-increasing rearrangement of $(k_0, \ldots, k_{d-1})$. The S_n-module $\mathcal{H}_{k_0,\ldots,k_{d-1}}$ decomposes into a direct sum of irreducible submodules. Fortunately, these submodules and their multiplicity are completely characterized by Young's rule. Moreover, below we shall describe how the decomposition into irreducible submodules of $\mathcal{H}_{k_0,\ldots,k_{d-1}}$ is related to, and determined by, the decomposition of $\mathcal{H}_\mu \equiv \mathcal{H}_{\mu_0,\ldots,\mu_{d-1}}$. (This allows us to explicitly identify links

between irreducible subspaces for the representation π.) Here the key combinatorial tool is the notion of a semistandard tableau.

We generalize the notion of λ-tableau, by saying that $T : [\lambda] \to \mathbb{N}$ is a *λ-tableau of type* $\mu = (\mu_0, ..., \mu_{d-1})$ if

$$\#\big\{c_{ij} : T(c_{ij}) = l\big\} = \mu_l \quad \text{for} \quad l = 0, \dots, d-1.$$

Then T is called *semistandard* if the numbers that T assigns to the cells of the diagram determined by λ are non-decreasing along rows and strictly increasing down columns. Let us fix a bijection $t_0 : [\lambda] \to \{1, \dots, n\}$. Then S_n acts on the sets $I(\lambda, \mu)$, the set of λ-tableau of type μ, *via*

$$\sigma(T) = T\, t_0^{-1} \sigma\, t_0 \quad \text{for} \quad \sigma \in S_n.$$

Given t_0, we will say that T_1 and T_2 are *row* (*column*) equivalent, and write $T_1 \sim_{t_0}^r T_2$, if $\sigma T_1 = \sigma T_2$ holds for all permutations σ in the row (respectively column) stabilizer of t_0. In particular, this means that T_1 and T_2 are row equivalent if and only if T_2 is obtained from T_1 by permuting the entries in each row accordingly.

In order to define the linking module maps we first need an appropriate bijection. We denote by $\mathcal{P}_{\mu_0,\dots,\mu_{d-1}}$ the set of partitions $(A_0, ..., A_{d-1})$ of $\{1, \dots, n\}$ such that $|A_l| = \mu_l$. Then $\mathcal{P}_{\mu_0,\dots,\mu_{d-1}}$ induces a natural relabelling of the standard basis for $\mathcal{H}_\mu$ by

$$f_{A_0,\dots,A_{d-1}} = |i_1 \cdots i_n\rangle \tag{3}$$

where $A_l = \{1 \leq j \leq n \mid i_j = l\}$ for $0 \leq l < d$. (Every n-tuple $(i_1, \dots, i_n)$ is associated with a unique d-tuple of sets $(A_0, \dots, A_{d-1})$ defined in this way.)

Next we define $\gamma_{t_0} : I(\lambda, \mu) \to \mathcal{P}_{\mu_0,\dots,\mu_{d-1}}$ by

$$\gamma_{t_0}(T) = (A_0, \dots, A_{d-1}),$$

where

$$A_l = \{1 \leq j \leq n \mid T t_0^{-1}(j) = l\} \quad \text{for} \quad 0 \leq l < d.$$

Every λ-tableau T of type μ induces an S_n-module map $\Theta_T : M^\lambda \to M^\mu$ by

$$\Theta_T(e_{\{t_0\}}) = \sum_{T' \sim^r_{t_0} T,\ \gamma_{t_0}(T')=(A_0,\dots,A_{d-1})} f_{A_0,\dots,A_{d-1}}.$$

Clearly this extends to an S_n-module homomorphism by defining

$$\Theta_T(e_{\{\sigma(t_0)\}}) = \sigma(\Theta_T(e_{\{t_0\}})).$$

This rather abstract description is in fact very concrete. Given indices $i_1,\dots,i_n \in \{0,\dots,d-1\}$ and a λ-tableau $t : [\lambda] \to \{1,\dots,n\}$ we form the generalized tableau $t_{|i_1\cdots i_n\rangle} : [\lambda] \to \{0,\dots,d-1\}$ by

$$t_{|i_1\cdots i_n\rangle}(c_{ij}) = i_{t0(c_{ij})}.$$

This means we write the entries $i_1,\dots,i_n$ into λ following the order given by t_0. Then we say that

$$(i_1,\dots,i_n) \sim_{t_0} (i'_1,\dots,i'_n)$$

if there exists a permutation $\sigma \in S_n$ such that $i'_j = i_{\sigma(j)}$ for $1 \le j \le n$ and $t_0^{-1}\sigma t_0$ leaves the rows of λ invariant. Therefore, we obtain

$$\Theta_T(e_{\{t_0\}}) = \sum_{(i_1,\dots,i_n)\sim_{t_0}\gamma_{t_0}(T)} |i_1\cdots i_n\rangle,$$

where here we identify $\gamma_{t_0}(T)$ with the n-tuple determined by the partition $\gamma_{t_0}(T) = (A_0,\dots,A_{d-1})$ as in (3).

For example, let $d = 3$, $n = 5$ and let $t_0 : [\lambda] \to \{1,\dots,5\}$ be given by

$$t_0 = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}$$

and $T : [\lambda] \to \{0,1,2\}$ be given by

$$T = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}.$$

This yields, by reading off the entries from the corresponding position in the diagram,

$$\gamma_{t_0}(T) = (A_0, A_1, A_2) = (\{1,3\},\{2,4\},\{5\}).$$

Following (3), $\gamma_{t_0}(T)$ is identified with $(i_1, i_2, i_3, i_4, i_5) = (0,1,0,1,2)$. Moreover, the list of equivalent indices is:

$$\Big\{(0,1,0,1,2),(0,1,1,0,2),(0,0,1,1,2),(1,0,0,1,2), \\ (1,0,1,0,2),(1,1,0,0,2)\Big\}.$$

Indeed, according to t_0 we have to fix the 5th coordinate and the other four vary in all possible ways. Thus we have

$$\Theta_T(e_{\{t\}}) = \sum_{(i_1,\ldots,i_n)\sim_t \gamma_t(T)} |i_1 \cdots i_n\rangle \quad \text{for} \quad e_{\{t\}} \in M^\lambda.$$

Following Young's rule (see chapter 2, **[18]**) we obtain:

Theorem 5.2. *Let $\mu = (k_0^*, \ldots, k_{d-1}^*)$ be the non-increasing rearrangement of $(k_0, \ldots, k_{d-1})$. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be such that $\lambda_1 \geq \ldots \geq \lambda_r$ and $\sum_i \lambda_i = n$. Then*

$$\mathcal{H}^\lambda_{k_0,\ldots,k_{d-1}} \equiv \operatorname{span}\left\{\Theta_T\Big(\sum_{\sigma\in C_t}(\operatorname{sgn}\sigma)e_{\sigma\{t\}}\Big) : T \in I(\lambda,\mu),\ t\ \lambda\text{-tableau}\right\}$$

is an irreducible S_n-submodule. The restriction of the representation π to $\mathcal{H}^\lambda_{k_0,\ldots,k_{d-1}}$ is equivalent to the irreducible representation π_μ of S_n on S^μ and has multiplicity

$$m = \#\big\{T : T \text{ semistandard } \lambda\text{-tableau of type } \mu\big\}.$$

If we collect all this information for all $(k_0, ..., k_{d-1})$, we can describe the full representation π of $\mathbb{C}[S_n]$:

Corollary 5.3. *Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be such that $\lambda_1 \geq \ldots \geq \lambda_r$ and $\sum_i \lambda_i = n$, and let P_λ be the projection of $\mathcal{H}_{d^n}$ onto*

$$\mathcal{H}^\lambda \equiv \bigoplus_{k_0,\ldots,k_{d-1}} \mathcal{H}^\lambda_{k_0,\ldots,k_{d-1}},$$

where the sum indexes over all $k_0, \ldots, k_{d-1}$ such that $\sum_{l=0}^{d-1} k_l = n$.

Then P_λ is the minimal central projection for $\pi(S_n)''$ which supports the irreducible submodule $\mathcal{S}^\lambda$. Moreover, $P_\lambda \pi P_\lambda$ is equivalent to the representation π_λ on $\mathcal{S}^\lambda$ with multiplicity

$$m_{\lambda,d} = \sum_{\mu_0 \geq \cdots \geq \mu_{d-1}} \mathrm{Arr}(\mu)\,\mathrm{SST}(\mu),$$

where

$$\mathrm{Arr}(\mu) = \#\Big\{(k_0, ..., k_{d-1}) : (k_0^*, ..., k_{d-1}^*) = (\mu_0, ..., \mu_{d-1})\Big\}$$

and

$$\mathrm{SST}(\mu) = \#\Big\{T : [\lambda] \to \{0, \dots, d-1\} \;\Big|\; T \textit{ semistandard } \lambda\textit{-tableau of type } \mu\Big\}.$$

In particular, for a maximal system $\mathcal{S}$,

$$\mathrm{Fix}(\mathcal{E}_S) = \mathcal{A}_\mathcal{S}' = \pi(S_n)'' \cong \sum_{m_{\lambda,d} \neq 0} \mathbb{M}_{\dim(\mathcal{S}^\lambda)} \otimes \mathbb{1}_{m_{\lambda,d}}$$

describes the representation in irreducible parts with multiplicity.

Corollary 5.4. *Let $\mathcal{S}$ be a maximal system, then $\mathcal{A}_\mathcal{S}$ is isomorphic to*

$$\mathcal{A}_\mathcal{S} \cong \sum_{m_{\lambda,d} \neq 0} \mathbb{M}_{m_{\lambda,d}} \otimes \mathbb{1}_{\dim \mathcal{S}^\lambda}$$

and the multiplicity of the component $\mathbb{M}_{m_{\lambda,d}}$ is given by $\dim \mathcal{S}^\lambda$.

Let us mention that $\mathcal{H}^\lambda$ may also be described by the so-called Garnier relations. Given $\lambda = (\lambda_1, \dots, \lambda_r)$, we fix the tableau T_λ such that $T_\lambda(c_{kj}) = k$ for all cells c_{kj} in $[\lambda]$. It follows that every index $i = (i_1, \dots, i_n)$ defines a tableau $T_i : [\lambda] \to \{0, \dots, d-1\}$ given by $T_i(c_{kj}) = i_{T_\lambda(c_{kj})}$. Let $G(J)$ be the collection of coset representatives $\{\nu X : \nu \in Y\}$, where Y is the subgroup of S_n which fixes every element outside both $C_h(T_\lambda) \cup J$ and $Y = X \cap C(T)$. Then as is proved in **[15]**(p.66, 5.2b), $|\psi\rangle \in \mathcal{H}^\lambda$ if and only if

(1) $\langle\psi||i\rangle = 0$ for all i such that T_i has equal entries in two distinct places in the same column.

(2) $\pi(\sigma)(|\psi\rangle) = \mathrm{sgn}(\sigma)|\psi\rangle$ for all σ in the column stabilizer of T_λ.

(3) $\sum_{\nu\in G(J)} \mathrm{sgn}(\nu)\pi(\nu^{-1})|\psi\rangle = 0$ for any non-empty set in the column stabilizer of $C_{h+1}(T_\lambda)$.

6. Examples

6.1. **The case $d = 2$ and general n.** As mentioned above, the case of $d = 2$ and general n was extensively examined in [**16**]. Let us indicate how this can be accomplished with Young tableaux.

When $d = 2$, we have the pairs (k_1, k_2) given by $(n - k, k)$ where $k = 0, ..., n$. In terms of λ-tableau we have to calculate $m_{\lambda,2}$. In terms of types we only have to consider diagrams $\mu_k = (n - k, k)$ where $0 \le k \le 2n$. But we have to be aware that every type allows combinations $(n-k, k)$ and $(k, n-k)$. Given $\lambda = (\lambda_1, ..., \lambda_r)$, we observe that to obtain a semistandard tableau, we must have $r = 2$. Indeed, we are forced to put 0's in the first row on the first λ_2 positions and 1's in the second row. Thus for fixed k, j with $2k \le n$ an $2j \le n$, we need $k \le j$ in order to produce a λ-tableau of type μ. Since, we also know that there are $n - k$ 0's, we do not have a choice and we have to put them all in the first row one after another. Thus for a fixed λ, we find

$$m_{(n-j,j),2} = \sum_{k\le j}^{\lfloor n/2\rfloor} 2 + 1 = 2(\lfloor n/2\rfloor - j) + 1$$

if n is even and

$$m_{(n-j,j),2} = 2(\lfloor n/2\rfloor - j)$$

if n is odd, where $\lfloor\cdot\rfloor$ denotes the greatest integer part of some number.

We also have to understand $\dim(\mathcal{S}^\lambda)$. If $\lambda = 0$, we get $\dim(\mathcal{S}^{(n,0)}) = 1$. If $1 \le j < \frac{n}{2}$, we see for cells c_{1l} with $l \le j$ the hook length is $1 + (n - j + 1 - i)$. This yields $n(n - 2j + 1)/j!(n - j + 1)!$ and hence

$$\dim(\mathcal{S}^{(n-j,j)}) = \begin{cases} 1 & \text{if } j = 0 \\ \frac{n-2j+1}{n+1}\binom{n+1}{j} & \text{if } 1 < j \le \frac{n}{2} \end{cases}.$$

Let us consider the examples $n = 4$ and $n = 5$. Then

$$\dim \mathcal{S}^{(4,0)} = 1, \quad \dim \mathcal{S}^{(3,1)} = 3, \quad \dim \mathcal{S}^{(2,2)} = 2$$

and

$$m_{(4,0),2} = 5, \quad m_{(3,1),2} = 3, \quad m_{(2,2),2} = 1.$$

In the case $n = 5$, we have

$$\dim \mathcal{S}^{(5,0)} = 1, \quad \dim \mathcal{S}^{(4,1)} = 4, \quad \dim \mathcal{S}^{(3,2)} = 5$$

and

$$m_{(5,0),2} = 6, \quad m_{(4,1),2} = 4, \quad m_{(3,2),2} = 2.$$

Bases for $\mathcal{H}_{2^n}$ which yield the associated algebra decompositions may be computed as well. Below we do this for a more intricate example.

6.2. **The case $d = 3$ and $n = 4$.** If $d = 3$ and $n = 4$, then the set of λ-diagrams which admit semistandard tableaux is given by

$$\left\{ (4) = \begin{array}{|c|c|c|c|}\hline x & x & x & x \\ \hline\end{array} \quad (3,1) = \begin{array}{|c|c|c|}\hline x & x & x \\ \hline x & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \quad (2,2) = \begin{array}{|c|c|}\hline x & x \\ \hline x & x \\ \hline\end{array} \quad (2,1,1) = \begin{array}{|c|c|}\hline x & x \\ \hline x & \multicolumn{1}{c}{} \\ \cline{1-1} x & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \right\}.$$

In this case, $\pi(S_4)$ acts on $\mathcal{H}_{d^n} = \mathcal{H}_{81}$. As in Theorem 5.1, $M^{(4)}$ is isomorphic to $\mathcal{H}_{4,0,0}$, $\mathcal{H}_{0,4,0}$ and $\mathcal{H}_{0,0,4}$; $M^{(2,2)}$ is isomorphic to $\mathcal{H}_{2,2,0}$, $\mathcal{H}_{2,0,2}$ and $\mathcal{H}_{0,2,2}$; $M^{(2,1,1)}$ is isomorphic to $\mathcal{H}_{2,1,1}$, $\mathcal{H}_{1,2,1}$ and $\mathcal{H}_{1,1,2}$; *etc*, so that the multiplicities for the M^λ are 3 for $M^{(4)}$, $M^{(2,2)}$ and $M^{(2,1,1)}$ and 6 for $M^{(3,1)}$. The dimensions of the Specht modules $\mathcal{S}^\lambda$ using the hook length formula are given by

$$\dim \mathcal{S}^{(4)} = 1, \quad \dim \mathcal{S}^{(3,1)} = 3, \quad \dim \mathcal{S}^{(2,2)} = 2, \quad \dim \mathcal{S}^{(2,1,1)} = 3.$$

Now, we have to compute the multiplicities of $\mathcal{S}^\lambda$ in M^μ. If $\mu = (4)(= (4,0,0))$, then the only semistandard tableau of type μ is $\lambda = (4)$ with 0 in each cell. Thus $M^{(4)} \cong \mathcal{S}^{(4)}$. Further, every M^μ supports a single

copy of $\mathcal{S}^{(4)}$ *via* the $\lambda = (4)$-tableau with cell entries given by μ. For $\mu = (3,1)$, the possible semistandard tableaux are

$$\begin{array}{|c|c|c|c|}\hline 0&0&0&1\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 0&0&0\\\hline 1\\\cline{1-1}\end{array}.$$

This gives

$$M^{(3,1)} \cong \mathcal{S}^{(4)} \oplus \mathcal{S}^{(3,1)}.$$

For $\mu = (2,2)$, we have

$$\begin{array}{|c|c|c|c|}\hline 0&0&1&1\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 0&0&1\\\hline 1\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 0&0\\\hline 1&1\\\hline\end{array}.$$

Thus we obtain

$$M^{(2,2)} \cong \mathcal{S}^{(4)} \oplus \mathcal{S}^{(3,1)} \oplus \mathcal{S}^{(2,2)}.$$

Finally, for $\mu = (2,1,1)$ we find

$$\begin{array}{|c|c|c|c|}\hline 0&0&1&2\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 0&0&1\\\hline 2\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 0&0&2\\\hline 1\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 0&0\\\hline 1&2\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 0&0\\\hline 1\\\cline{1-1} 2\\\cline{1-1}\end{array}.$$

This means

$$M^{(2,1,1)} \cong \mathcal{S}^{(4)} \oplus (\mathcal{S}^{(3,1)} \otimes \mathbb{1}_2) \oplus \mathcal{S}^{(2,2)} \oplus \mathcal{S}^{(2,1,1)}.$$

Putting this all together, we find the module decomposition of $\pi(S_4)$ is given by

$$\begin{aligned}
\pi(S_4) &= (M^{(4)} \otimes \mathbb{1}_3) \oplus (M^{(3,1)} \otimes \mathbb{1}_6) \oplus (M^{(2,2)} \otimes \mathbb{1}_3) \oplus (M^{(2,1,1)} \otimes \mathbb{1}_3) \\
&= (\mathcal{S}^{(4)} \otimes \mathbb{1}_3) \oplus \big((\mathcal{S}^{(4)} \oplus \mathcal{S}^{(3,1)}) \otimes \mathbb{1}_6\big) \qquad (4)\\
&\oplus \big((\mathcal{S}^{(4)} \oplus \mathcal{S}^{(3,1)} \oplus \mathcal{S}^{(2,2)}) \otimes \mathbb{1}_3\big) \qquad (5)\\
&\oplus \big((\mathcal{S}^{(4)} \oplus (\mathcal{S}^{(3,1)} \otimes \mathbb{1}_2) \oplus \mathcal{S}^{(2,2)} \oplus \mathcal{S}^{(2,1,1)}) \otimes \mathbb{1}_3\big) \qquad (6)\\
&= (\mathcal{S}^{(4)} \otimes \mathbb{1}_{15}) \oplus (\mathcal{S}^{(3,1)} \otimes \mathbb{1}_{15}) \oplus (\mathcal{S}^{(2,2)} \otimes \mathbb{1}_6) \oplus (\mathcal{S}^{(2,1,1)} \otimes \mathbb{1}_3). \qquad (7)
\end{aligned}$$

The direct sums in (4), (5) and (6) are understood to be 'linked', as reflected in (7). It now follows that

$$\begin{aligned}
\mathrm{Fix}(\mathcal{E}_{\mathcal{S}}) &= \pi(S_4)'' \\
&\cong (\mathbb{C} \otimes \mathbb{1}_{15}) \oplus (\mathbb{M}_3 \otimes \mathbb{1}_{15}) \oplus (\mathbb{M}_2 \otimes \mathbb{1}_6) \oplus (\mathbb{M}_3 \otimes \mathbb{1}_3). \qquad (8)
\end{aligned}$$

Notice also that $\mathbb{M}_3$ is the largest full matrix algebra which can be injected into $\mathrm{Fix}(\mathcal{E}_{\mathcal{S}})$ as a subalgebra.

Let us now describe the bases for the decomposition

$$\begin{aligned}\mathcal{H}_{3^4} &= (\mathcal{H}_{4,0,0} \oplus \mathcal{H}_{0,4,0} \oplus \mathcal{H}_{0,0,4}) \oplus (\mathcal{H}_{2,1,1} \oplus \mathcal{H}_{1,2,1} \oplus \mathcal{H}_{1,1,2}) \\ &\quad \oplus (\mathcal{H}_{3,1,0} \oplus \mathcal{H}_{3,0,1} \oplus \mathcal{H}_{0,3,1} \oplus \mathcal{H}_{1,3,0} \oplus \mathcal{H}_{0,1,3} \oplus \mathcal{H}_{1,0,3})\end{aligned}$$

which yields this algebra decomposition. This is easy for $\lambda = (4)$. Indeed, for every $\mathcal{H}_{k_0,k_1,k_2}$ this is given by the invariant average vector

$$h^{(4)} = \sum_{k_0(i_1,i_2,i_3,i_4)=k_0,\ldots,k_2(i_1,i_2,i_3,i_4)=k_2} |i_1 i_2 i_3 i_4\rangle.$$

In the following we will only discuss the case where $k_0 \geq k_1 \geq k_2$ (*i.e.*, $\mathcal{H}_{4,0,0}$, $\mathcal{H}_{3,1,0}$, $\mathcal{H}_{2,1,1}$). For $\lambda = \mu$, we have a natural embedding $\mathcal{S}^\lambda \subseteq M^\lambda \cong \mathcal{H}_{k_0,k_1,k_2}$ given by

$$h_t = \sum_{\sigma \in C_t} \mathrm{sgn}(\sigma) \sum_{(i_1,\ldots,i_4) \sim_t (i_1^t,\ldots,i_4^t)} |i_1 i_2 i_3 i_4\rangle, \tag{9}$$

for all λ-tableau t of type μ. Let us illustrate this in our examples. If $\lambda = (3,1)$, we have 3 standard tableaux

$$t_0 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad t_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad t_2 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}.$$

The column stabilizer of t_0, t_1, t_2 is $C_{t_0} = \{1, (14)\}$, $C_{t_1} = \{1, (13)\}$, $C_{t_2} = \{1, (12)\}$. The space $\mathcal{H}_{(3,1)}$ has the basis

$$|0001\rangle, \quad |0010\rangle, \quad |0100\rangle, \quad |1000\rangle.$$

Now, we define on $\mathcal{H}_{(3,1)}$

$$A_{t_i} = \sum_{\sigma \in C_{t_i}} \mathrm{sgn}(\sigma)\pi(\sigma).$$

The range of A_{t_i} is given by the vectors

$$\begin{aligned}h_{t_0} &= |0001\rangle - |1000\rangle, \\ h_{t_1} &= |0010\rangle - |1000\rangle, \\ h_{t_2} &= |0100\rangle - |1000\rangle.\end{aligned}$$

This provides us with the basis for

$$M^{(3,1)} \cong \mathcal{H}_{(3,1)} = \mathrm{span}\{h^{(4)}\} \oplus \mathrm{span}\{h_{t_0}, h_{t_1}, h_{t_2}\}.$$

Now, we consider $\mathcal{H}_{(2,2,0)}$ spanned by

$$|0011\rangle, \quad |0110\rangle, \quad |0101\rangle, \quad |1100\rangle, \quad |1010\rangle, \quad |1001\rangle.$$

For $\lambda = (3,1)$ we have the following list of λ-tableaux of type $(2,2)$

$$\begin{array}{|c|c|c|}\hline 0&0&1\\\hline 1\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 0&1&1\\\hline 0\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 0&1&0\\\hline 1\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&1&0\\\hline 0\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&0&1\\\hline 0\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&0&0\\\hline 1\\\cline{1-1}\end{array}.$$

Here we used $t_0 = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}$. Only the first tableaux is semistandard and yields an injection

$$\Theta \equiv \Theta_{\begin{array}{|c|c|c|}\hline 0&0&1\\\hline 1\\\cline{1-1}\end{array}} : \mathcal{S}^{(3,1)} \to \mathcal{H}_{(2,2)}$$

with

$$\Theta(e_{\{t_0\}}) = \sum_{(i_1,\ldots,i_4)\sim_{t_0}(0,0,1,1)} |i_1 i_2 i_3 i_4\rangle.$$

This means

$$\Theta(e_{\{t_0\}}) = |0011\rangle + |0101\rangle + |1001\rangle.$$

(See the example in the last section for $n = 5$.) Further, this vector is a cyclic vector for the image of $\mathcal{S}^{(3,1)}$ in $\mathcal{H}_{(2,2)}$. The polytabloid is

$$e_{t_0} = \sum_{\sigma \in C_{t_0}} \mathrm{sgn}(\sigma)\sigma e_{\{t_0\}} = e_{\{t_0\}} - (14)e_{\{t_0\}}.$$

Therefore $\Theta(\mathcal{S}^{(3,1)})$ is the module generated by

$$\Theta(e_{t_0}) = h_{t_0} = |0011\rangle - |1010\rangle + |0101\rangle - |1100\rangle.$$

Equivalently,

$$\begin{aligned}
\Theta(\mathcal{S}^{(3,1)}) &= \mathrm{span}\{h_{t_0}, (12)h_{t_0}, (13)h_{t_0}\} \\
&= \mathrm{span}\big\{|0011\rangle - |1010\rangle + |0101\rangle - |1100\rangle, \\
&\quad |1100\rangle - |1001\rangle + |0110\rangle - |0011\rangle, \\
&\quad |0110\rangle - |0101\rangle + |1010\rangle - |1001\rangle\big\}.
\end{aligned}$$

Another way to find a basis is to consider $t_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$. In this case, $C_{t_1} = \{1, (13)\}$,

$$\Theta(e_{\{t_1\}}) = |0011\rangle + |0110\rangle + |1010\rangle$$

and

$$h_{t_1} = \Theta(e_{\{t_1\}}) - (13)\Theta(e_{\{t_1\}}) = |0011\rangle - |1001\rangle + |0110\rangle - |1100\rangle.$$

Similarly for $t_2 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$, we have $C_{t_1} = \{1, (12)\}$ and

$$\Theta(e_{\{t_2\}}) = |0101\rangle + |0110\rangle + |1100\rangle$$

and

$$h_{t_2} = \Theta(e_{\{t_2\}}) - (12)\Theta(e_{\{t_2\}}) = |0101\rangle - |1001\rangle + |0110\rangle - |1010\rangle.$$

The copy of $\mathcal{S}^{(2,2)}$ in $\mathcal{H}_{(2,2)}$ is again easy to find. We recall that $\mathcal{S}^{(2,2)}$ is spanned by the standard tableaux $\{e_{s_0}, e_{s_1}\}$ where

$$s_0 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \qquad s_1 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}.$$

The column stabilizers are given by $C_{s_0} = \{1, (13), (24), (13)(24)\}$ and by $C_{s_1} = \{1, (12), (34), (12)(34)\}$. This yields operators on $\mathcal{H}_{(2,2)}$,

$$A_{s_0} = 1 - \pi((13)) - \pi((24)) + \pi((13)(24))$$

and

$$A_{s_1} = 1 - \pi((12)) - \pi((34)) + \pi((12)(34)).$$

Applied to the unit vectors, we find the ranges

$$\mathrm{Ran}(A_{s_0}) = |0110\rangle - |1010\rangle - |0101\rangle + |1001\rangle$$

and

$$\mathrm{Ran}(A_{s_1}) = |0011\rangle - |1001\rangle - |0110\rangle + |1100\rangle.$$

Finally, we consider $\mathcal{H}_{(2,1,1)}$ with basis

$$\big\{|0012\rangle, |0021\rangle, |0102\rangle, |0120\rangle, |0201\rangle, |0210\rangle, |1002\rangle, |1020\rangle, |1200\rangle,$$
$$|2001\rangle, |2010\rangle, |2100\rangle\big\}.$$

The representation of $\mathcal{S}^{(4)}$ is 1-dimensional, given by the average of all these vectors. There are two copies of $\mathcal{S}^{(3,1)}$ corresponding to the two $(3,1)$-tableaux of type $(2,1,1)$

$$T = \begin{array}{|c|c|c|}\hline 0 & 0 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad T' = \begin{array}{|c|c|c|}\hline 0 & 0 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}.$$

The basis for $\mathcal{S}^{(3,1)}$ is given by $\{e_{t_0}, e_{t_1}, e_{t_2}\}$ where

$$t_0 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \quad t_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \quad t_2 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}.$$

Following the definition of $\Theta_T(e_{t_j})$, we get

$$\gamma_{t_0}(T) = (0,0,1,2), \quad \gamma_{t_1}(T) = (0,0,2,1), \quad \gamma_{t_2}(T) = (0,2,0,1).$$

Using row equivalence, we are allowed to permute the entries $\{1,2,3\}$ for t_0, the entries $\{1,2,4\}$ for t_1 and $\{1,3,4\}$ for t_2 and thus

$$\Theta_T(e_{\{t_0\}}) = |0012\rangle + |0102\rangle + |1002\rangle,$$
$$\Theta_T(e_{\{t_1\}}) = |0021\rangle + |0120\rangle + |1020\rangle,$$
$$\Theta_T(e_{\{t_2\}}) = |0201\rangle + |0210\rangle + |1200\rangle.$$

For t_0, t_1, t_2 we have to apply, respectively, $A_{T,t_0} = \mathbb{1} - \pi(14)$, $A_{T,t_1} = \mathbb{1} - \pi(13)$ and $A_{T,t_2} = \mathbb{1} - \pi(12)$ in order to obtain the image of the polytabloids:

$$h_{T,t_0} = |0012\rangle - |2010\rangle + |0102\rangle - |2100\rangle + |1002\rangle - |2001\rangle,$$
$$h_{T,t_1} = |0021\rangle - |2001\rangle + |0120\rangle - |2100\rangle + |1020\rangle - |2010\rangle,$$
$$h_{T,t_2} = |0201\rangle - |2001\rangle + |0210\rangle - |2010\rangle + |1200\rangle - |2100\rangle.$$

This is our first copy of $\mathcal{S}^{(3,1)}$. For the second, we exercise the same procedure in the case of T'.

$$\Theta_{T'}(e_{\{t_0\}}) = |0021\rangle + |0201\rangle + |2001\rangle,$$
$$\Theta_{T'}(e_{\{t_1\}}) = |0012\rangle + |0210\rangle + |2010\rangle,$$
$$\Theta_{T'}(e_{\{t_2\}}) = |0102\rangle + |0120\rangle + |2100\rangle.$$

This provides us with

$$h_{T',t_0} = |0021\rangle - |1020\rangle + |0201\rangle - |1200\rangle + |2001\rangle - |1002\rangle,$$

$$h_{T',t_1} = |0012\rangle - |1002\rangle + |0210\rangle - |1200\rangle + |2010\rangle - |1020\rangle,$$

$$h_{T',t_2} = |0102\rangle - |1002\rangle + |0120\rangle - |1020\rangle + |2100\rangle - |1200\rangle.$$

We have one copy of $\mathcal{S}^{(2,2)}$ which is spanned by

$$s_0 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \qquad s_1 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}.$$

Our $(2,2)$ tableau of type $(2,1,1)$ is given by $T = \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & 2 \\ \hline \end{array}$. This gives

$$\Theta(e_{\{s_0\}}) = |0012\rangle + |0021\rangle \quad \text{and} \quad \Theta(e_{\{s_1\}}) = |0102\rangle + |0201\rangle.$$

The operator is $\mathbb{1} - \pi(13) - \pi(24) + \pi((13)(24))$, determined by $C_{s_0} = \{1, (13), (24), (13)(24)\}$, and thus

$$h_{s_0} = |0012\rangle - |1002\rangle - |0210\rangle + |1200\rangle + |0021\rangle - |2001\rangle - |0120\rangle + |2100\rangle$$

and similarly for s_1 we apply $\mathbb{1} - \pi(12) - \pi(34) + \pi((12)(34))$ to obtain

$$h_{s_1} = |0102\rangle - |1002\rangle - |0120\rangle + |1020\rangle + |0201\rangle - |2001\rangle - |0210\rangle + |2010\rangle.$$

Finally we consider the copy of $\mathcal{S}^{(2,1,1)}$, which has basis $\{e_{r_0}, e_{r_1}, e_{r_2}\}$ where

$$r_0 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad r_1 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad r_2 = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

Here $T = \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$. This yields

$$\Theta_T(e_{\{r_0\}}) = |0012\rangle, \quad \Theta_T(e_{\{r_1\}}) = |0102\rangle, \quad \Theta_T(e_{\{r_2\}}) = |0120\rangle.$$

The column stabilizer of r_2 is given by all permutations which leave $\{1,2,3\}$ invariant. This yields

$$h_{r_2} = |0120\rangle - |0210\rangle - |1020\rangle + |1200\rangle + |2010\rangle - |2100\rangle.$$

Similarly, we have to look for all permutations of $\{1,3,4\}$ in the column stabilizer of r_0 and we obtain

$$h_{r_0} = |0012\rangle - |0021\rangle - |1002\rangle + |1020\rangle + |2001\rangle - |2010\rangle.$$

For the column stabilizer of r_1, we may permute $\{1,2,4\}$ and hence

$$h_{r_1} = |0102\rangle - |0201\rangle - |1002\rangle + |1200\rangle + |2001\rangle - |2100\rangle.$$

By equation (8), the largest full matrix algebra $\mathbb{M}_k$ that can be injected into the noise commutant here is $\mathbb{M}_3$, identified with the subalgebras of $\mathcal{A}'_{\mathcal{S}}$ unitarily equivalent to either $\mathbb{1}_{15} \otimes \mathbb{M}_3$ or $\mathbb{1}_3 \otimes \mathbb{M}_3$. Let us explicitly identify the copy of $\mathbb{1}_3 \otimes \mathbb{M}_3$. The set $\{h_{r_0}, h_{r_1}, h_{r_2}\}$ yields the copy of $S^{(2,1,1)}$ inside $\mathcal{H}^{(2,1,1)}$. A similar analysis yields the basis for the copy of $S^{(2,1,1)}$ inside $\mathcal{H}^{(1,2,1)}$. It is generated by $T' = \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$ and in this case

$$\Theta_{T'}(e_{\{r_0\}}) = |0112\rangle + |1012\rangle,$$

$$\Theta_{T'}(e_{\{r_1\}}) = |0112\rangle + |1102\rangle,$$

$$\Theta_{T'}(e_{\{r_2\}}) = |0121\rangle + |1120\rangle.$$

Thus we have

$$\left\{ \begin{array}{lcl} h'_{r_0} & = & |0112\rangle - |0121\rangle - |1102\rangle + |1120\rangle + |2101\rangle - |2110\rangle \\ h'_{r_1} & = & |0112\rangle - |0211\rangle - |1012\rangle + |1210\rangle + |2011\rangle - |2110\rangle \\ h'_{r_2} & = & |0121\rangle - |0211\rangle - |1021\rangle + |1201\rangle + |2011\rangle - |2101\rangle \end{array} \right. .$$

Further, the basis for the copy of $S^{(2,1,1)}$ inside $\mathcal{H}^{(1,1,2)}$ is generated by $T'' = \begin{array}{|c|c|}\hline 0 & 2 \\ \hline 1 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$ and in this case

$$\Theta_{T''}(e_{\{r_0\}}) = |0212\rangle + |2012\rangle,$$

$$\Theta_{T''}(e_{\{r_1\}}) = |0122\rangle + |2102\rangle,$$

$$\Theta_{T''}(e_{\{r_2\}}) = |0122\rangle + |2120\rangle.$$

Thus we have

$$\left\{ \begin{array}{lcl} h''_{r_0} & = & |0212\rangle - |0221\rangle - |1202\rangle + |1220\rangle + |2201\rangle - |2210\rangle \\ h''_{r_1} & = & |0122\rangle - |0221\rangle - |1022\rangle + |1220\rangle + |2021\rangle - |2120\rangle \\ h''_{r_2} & = & |0122\rangle - |0212\rangle - |1022\rangle + |1202\rangle + |2012\rangle - |2102\rangle \end{array} \right. .$$

Let $P_{(2,1,1)}$ be the projection of $\mathcal{H}$ onto the span of $\{h_{r_i}, h'_{r_j}, h''_{r_k} : 0 \leq i, j, k \leq 2\}$. Then $P_{(2,1,1)}$ is a minimal central projection for $\mathcal{A}'_{\mathcal{S}}$ and the 'compression subalgebra' $P_{(2,1,1)}\mathcal{A}'_{\mathcal{S}}P_{(2,1,1)} = \mathcal{A}'_{\mathcal{S}}P_{(2,1,1)} \subset \mathcal{A}'_{\mathcal{S}}$ is unitarily equivalent to $\mathbb{1}_3 \otimes \mathbb{M}_3$. In fact, with respect to the ordered basis

$$\{h_{r_0}, h_{r_1}, h_{r_2}, h'_{r_0}, h'_{r_1}, h'_{r_2}, h''_{r_0}, h''_{r_1}, h''_{r_2}\}$$

for $P_{(2,1,1)}\mathcal{H}$, we have the matrix representations

$$\mathcal{A}'_{\mathcal{S}}P_{(2,1,1)} = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix} : A \in \mathbb{M}_3 \right\}.$$

Note that the subspaces spanned by the sets $\{h_{r_i}\}$, $\{h'_{r_i}\}$ and $\{h''_{r_i}\}$ are perpendicular, but the vectors within each of these sets do not form an orthogonal basis for the corresponding subspace.

7. Conclusion

We have investigated the operator algebras of fixed points for a class of quantum channels we call universal collective rotation channels $\mathcal{E}_{\mathcal{S}}$. This class includes as a subclass the well-known class of collective rotation channels. We showed that such channels always have an abundance of noiseless subsystems and gave a method for explicitly computing them. In particular, the Young tableaux machine gives a clean approach for this process. In lower dimensional cases (*e.g.* when $d = 2$), our approach is more technical when compared to others in the literature (for instance [**16**]). However, an important advantage of the Young tableaux approach for higher dimensional cases is that it is particularly amenable to computations.

An issue we have not pursued here concerns the channels generated by non-maximal sets $\mathcal{S}$. The d^n-dimensional representations of J_x, J_y, J_z considered in [**16**] provide such an example, but we would expect there to be other interesting non-trivial examples of channels

$\mathcal{E}_{\mathcal{S}}$ for non-maximal $\mathcal{S}$. We emphasize that even for non-maximal $\mathcal{S}$ there is an abundance of noiseless subsystems because $\mathrm{Fix}(\mathcal{E}_{\mathcal{S}})$ contains $\pi(S_n)''$.

We also wonder what other representations of S_n correspond to physically meaningful unital channels, beyond π and its subrepresentations (which correspond to the compressions of ucr-channels). The recent preprint **[3]** of Bacon, et al, appears to present further insights into this topic, and also shows how the unitary base change from the standard basis to the basis given by the Young tableaux can be efficiently computed using quantum circuits.

Acknowledgements. We are grateful to the referee for helpful comments. We would like to thank John Holbrook, Raymond Laflamme and David Poulin for enlightening conversations. The first author was partially supported by NSF grant DMS 03-01116. The second and third authors were partially supported by NSERC grants. The third author also gratefully acknowledges support from the Institute for Quantum Computing and the Perimeter Institute for Theoretical Physics.

References

[1] D. Aharonov, M. Ben-Or, *Fault-tolerant quantum computation with constant error,* In Proc. 29th. Ann. ACM Symp. on Theory of Computing, page 176, New York, 1998, ACM. arxiv.org/quant-ph/9906129, quant-ph/9611025.

[2] W. Arveson, *An invitation to* C*-*algebras*, Graduate Texts in Mathematics, No. 39, Springer-Verlag, New York-Heidelberg, 1976.

[3] D. Bacon, I.L. Chuang, A.W. Harrow, *Efficient quantum circuits for Schur and Clebsch-Gordon transforms*, arxiv.org/quant-ph/0407082.

[4] S.D. Bartlett, T. Rudolph, R.W. Spekkens, *Decoherence-full subsystems and the cryptographic power of a private shared reference frame*, arXiv.org/quant-ph/0403161.

[5] S.D. Bartlett, T. Rudolph, R.W. Spekkens, *Classical and quantum communication without a shared reference frame*, Phys. Rev. Lett. **91**, 027901 (2003).

[6] J.-C. Boileau, D. Gottesman, R. Laflamme, D. Poulin, R.W. Spekkens, *Robust polarization-based quantum key distribution over collective-noise channel*, arXiv.org/quant-ph/0306199.

[7] P. Busch, J. Singh, *Luders theorem for unsharp quantum effects,* Phys. Lett. A, **249** (1998), 10-24.

[8] M.D. Choi, *Completely positive linear maps on complex matrices,* Lin. Alg. Appl. **10** (1975), 285-290.
[9] C. Cohen-Tannoudji, B. Diu, F. Laloe, *Quantum Mechanics, Volume One & Two*, John Wiley & Sons, Toronto, 1977.
[10] K.R. Davidson, C*-*algebras by example*, Fields Institute Monographs, **6**, Amer. Math. Soc., Providence, 1996.
[11] L.-M. Duan, G.-C. Guo, *Preserving coherence in quantum computation by pairing quantum bits*, Phys. Rev. Lett. **79** (1997), 1953.
[12] S. De Filippo, *Quantum computation using decoherence-free states of the physical operator algebra,* Phys. Rev. A **62**, 052307 (2000).
[13] E.M. Fortunato, L. Viola, M.A. Pravia, E. Knill, R. Laflamme, T.F. Havel, D.G. Cory, *Exploring noiseless subsystems via nuclear magnetic resonance,* Phys. Rev. A **67**, 062303 (2003).
[14] W. Fulton, J. Harris, *Representation theory, a first course*, Springer-Verlag New York, 1991.
[15] J.A. Green, *Polynomial representations of* GL_n, Lecture Notes in Mathematics 830, Springer-Verlag, New York, 1980.
[16] J.A. Holbrook, D.W. Kribs, R. Laflamme, D. Poulin, *Noiseless subsystems for collective rotation channels in quantum information theory*, Integral Equations & Operator Theory, to appear.
[17] J.A. Holbrook, D.W. Kribs, R. Laflamme, *Noiseless subsystems and the structure of the commutant in quantum error correction*, Quantum Information Processing **2** (2003), 381-419.
[18] G. James, A. Kerber, *The representation theory of the symmetric group,* Encyclopedia of Mathematics and its Applications, Addison-Wesley Publishing Company, Toronto, 1981.
[19] M. Junge, *Noncommutative Poisson random measure,* preprint, 2003.
[20] J. Kempe, D. Bacon, D.A. Lidar, K.B. Whaley, *Theory of decoherence-free fault-tolerant universal quantum computation,* Phys. Rev. A **63**, 042307 (2001).
[21] E. Knill, R. Laflamme, A. Ashikhmin, H. Barnum, L. Viola, W.H. Zurek, *Introduction to Quantum Error Correction*, Los Alamos Science, November 27, 2002.
[22] E. Knill, R. Laflamme, L. Viola, *Theory of quantum error correction for general noise*, Phys. Rev. Lett. **84** (2000), 2525-2528.
[23] E. Knill, R. Laflamme, W. H. Zurek, *Resilient quantum computation: error models and thresholds,* Science **279** (1998), 342-345.
[24] E. Knill, R. Laflamme, *A theory of quantum error-correcting codes,* Phys. Rev. A **55** (1997), 900.
[25] K. Kraus, *States, Effects and Operations: Fundamental Notions of Quantum Theory,* Lecture Notes in Physics, vol. 190, Berlin: Springer-Verlag, 1983.
[26] K. Kraus, *General state changes in quantum theory,* Ann. Physics **64** (1971), 311-335.
[27] D.W. Kribs, *A quantum information theory primer for operator theorists*, preprint, 2004.
[28] D.W. Kribs, *Quantum channels, wavelets, dilations, and representations of* $\mathcal{O}_n$, Proc. Edin. Math. Soc., **46** (2003).
[29] R. Laflamme, private communication.

[30] D. A. Lidar, I. L. Chuang, K. B. Whaley, *Decoherence free subspaces for quantum computation*, Phys. Rev. Lett. **81** (1998), 2594.

[31] M.A. Nielsen, I.L. Chuang, *Quantum computation and quantum information*, Cambridge University Press, 2000.

[32] V. Paulsen, *Completely bounded maps and operator algebras,* Cambridge University Press, Cambridge, United Kingdom, 2002.

[33] J. Preskill, *Reliable quantum computers,* Proc. R. Soc. Lond. A, **454** (1998), 385-410.

[34] W.F. Stinespring, *Positive functions on* C^**-algebras*, Proc. Amer. Math. Soc. **6** (1955), 211-216.

[35] M. Takesaki, *Theory of operator algebras I*, Springer-Verlag, New York-Heidelberg, 1979.

[36] L. Viola, E. Knill, R. Laflamme, *Constructing qubits in physical systems*, J. Phys. A **34**, 7067 (2001).

[37] L. Viola, E.M. Fortunato, M.A. Pravia, E. Knill, R. Laflamme, D.G. Cory, *Experimental realization of noiseless subsystems for quantum information processing,* Science **293**, 2059 (2001).

[38] P. Zanardi, S. Lloyd, *Topological protection and quantum noiseless subsystems*, Phys. Rev. Lett. **90**, 067902 (2003).

[39] P. Zanardi, *Stabilizing quantum information*, Phys. Rev. A **63**, 012301 (2001).

[40] P. Zanardi, M. Rasetti, *Noiseless quantum codes*, Phys. Rev. Lett. **79** (1997), 3306.

[41] V. Zarikian, *Algorithms for operator algebra calculations*, Lin. Alg. Appl., to appear.

[1]Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, Illinois, USA 61801-2975

[2]Department of Mathematics and Statistics, University of Guelph, Guelph, ON, CANADA N1G 2W1.

[3]Institute for Quantum Computing, University of Waterloo, Waterloo, ON, CANADA N2L 3G1.

[4]Perimeter Institute for Theoretical Physics, 35 King St. North, Waterloo, ON, CANADA N2J 2W9.

E-mail address: `junge@math.uiuc.edu, pkim@uoguelph.ca, dkribs@uoguelph.ca`